The transformations of non-abelian gauge fields under translations

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I consider infinitesimal translations $x'^{\alpha} = x^{\alpha} + \delta x^{\alpha}$ and demand that Noether's approach gives a symmetric energy-momentum tensor as it is required for gravitational sources. This argument determines the transformations of non-abelian gauge fields under infinitesimal translations to differ from the usually assumed invariance by the gauge transformation, ${A'^a}_\gamma(x') - {A^a}_\gamma(x) = \partial_\gamma[\delta x_\beta\, A^{a\,\beta}(x)] + {C^a}_{bc}\,\delta x_\beta\, A^{c\,\beta}(x)\, {A^b}_\gamma(x)$ where the ${C^a}_{bc}$ are the structure constants of the gauge group.

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In a previous paper [1] I have determined the transformations of the electromagnetic potentials under translations from the requirement that the energy-momentum tensor as it comes out of Noether's theorem [2,3] ought to be symmetric. Such a result is desireable, because the energy-momentum tensor enters as source of the gravitational field and the symmetry transformations of general covariance yield a symmetric tensor, see for instance [4,5]. A more detailed motivation is given in my first paper. Here I extend the argument to non-abelian gauge theories.

Following the notation of Weinberg's book [6], the Lagrange density is

$$\mathcal{L} = -\frac{1}{4}\, {F^a}_{\alpha\beta}\, F^{a\,\alpha\beta} \tag{1}$$

with

$${F^a}_{\alpha\beta} = \partial_\alpha {A^a}_\beta - \partial_\beta {A^a}_\alpha + {C^a}_{bc}\, {A^b}_\alpha\, {A^c}_\beta \tag{2}$$

where the ${C^a}_{bc}$ are the structure constants of the gauge group. The Lagrangian (1) is invariant under the gauge transformations of the fields:

$${A^a}_\alpha \mapsto {A^a}_\alpha + \partial_\alpha \epsilon^a(x) + {C^a}_{bc}\, \epsilon^c(x)\, {A^b}_\alpha\ . \tag{3}$$

As in [1] the observation is that the conventionally assumed invariance of the gauge fields under translations

$$x'^{\,\alpha} = x^\alpha + \delta x^\alpha \tag{4}$$

can be enlarged by gauge transformations and the general form reads

$${A'^a}_\gamma(x') = {A^a}_\gamma(x) + \partial_\gamma \epsilon^a(x) + {C^a}_{bc}\, \epsilon^c(x)\, {A^b}_\gamma\ . \tag{5}$$

Repeating the arguments of Noether's theorem in the version of [3] and requesting a symmetric energy-momentum tensor determines the gauge transformation uniquely and leads to the transformation law stated in the abstract

$$\begin{aligned} {A'^a}_\gamma(x') = {A^a}_\gamma(x) + \partial_\gamma\left[\delta x_\beta\, A^{a\,\beta}(x)\right] \\ + {C^a}_{bc}\, \delta x_\beta\, A^{c\,\beta}(x)\, {A^b}_\gamma(x)\ . \end{aligned} \tag{6}$$

The remainder of this letter is devoted to the derivation of this equation and my treatment follows closely [1] where also a few additional steps can be found.

First, let us consider general fields ψ_k and recall the derivation of the relativistic Euler Lagrange equations from the action principle. The action is a four dimensional integral over a scalar Lagrangian density

$$\mathcal{A} = \int d^4x\, \mathcal{L}(\psi_k,\, \partial_\alpha \psi_k)\ . \tag{7}$$

Variations of the fields are defined as functions

$$\delta\psi_k(x) = \psi'_k(x) - \psi_k(x) \tag{8}$$

which are non-zero for some localized space-time region. The action is required to vanish under such variations

$$0 = \delta\mathcal{A} =$$
$$\sum_k \int d^4x \left[(\delta\psi_k)\, \frac{\partial \mathcal{L}}{\partial \psi_k} + (\delta\, \partial_\alpha \psi_k)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_k)} \right]\ . \tag{9}$$

Integration by parts allows to factor $\delta\psi_k$ out and, because all the $\delta\psi_k$ are independent, we arrive at the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \psi_k} - \partial_\alpha\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_k)} = 0\ . \tag{10}$$

Together with the anti-symmetry of ${F^a}_{\alpha\beta}$ in the Lorentz indices, the Euler-Lagrange equations imply the relation

$$\partial_\gamma\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha {A^a}_\gamma)} = {C^b}_{ac}\, {A^c}_\beta\, \frac{\partial \mathcal{L}}{\partial(\partial_\beta {A^b}_\alpha)}\ . \tag{11}$$

Noether's theorem applies to transformations of the coordinates for which the transformations of the field functions are also known and we introduce, in addition to (8), a second type of variations which combines space-time and their corresponding field variations

$$\overline{\delta}\psi_k(x) = \psi'_k(x') - \psi_k(x)\,. \tag{12}$$

Using

$$\psi'_k(x') = \psi'_k(x) + \delta x^\alpha\ \partial_\alpha \psi_k(x)$$

we find a relation between the variations (12) and (8)

$$\overline{\delta}\psi_k(x) = \delta\psi_k(x) + \delta x^\alpha \ \partial_\alpha \psi_k(x)\,. \tag{13}$$

For a scalar field ψ symmetry under translations means

$$\overline{\delta}\psi(x) = \psi'(x') - \psi(x) = 0\,. \tag{14}$$

But for the gauge fields we allow (5)

$$\begin{aligned}\overline{\delta}A^a{}_\gamma(x) &= A'^a{}_\gamma(x') - A^a{}_\gamma(x) \\ &= \partial_\gamma \epsilon^a(x) + C^a_{\ bc}\, \epsilon^c(x)\, A^b{}_\gamma(x)\end{aligned} \tag{15}$$

and equation (13) becomes

$$\delta A^a{}_\gamma = \partial_\gamma \epsilon^a(x) + C^a_{\ bc}\, \epsilon^c(x)\, A^b{}_\gamma - \delta x^\alpha \ \partial_\alpha A^a{}_\gamma(x)\ . \tag{16}$$

As the Lagrange density is a scalar, we get for its combined variation (12)

$$0 = \overline{\delta}\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \delta\mathcal{L} + \delta x^\alpha \ \partial_\alpha \mathcal{L} \tag{17}$$

where besides (14) we used the relation (13). Our aim is to factor an over-all variation δx^α out. For $\delta\mathcal{L}$ we proceed as in equation (9), where the ψ_k fields are now replaced by the gauge fields $A^a{}_\gamma$

$$\delta\mathcal{L} = (\delta A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial A^a{}_\gamma} + (\delta \partial_\alpha A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)}\,.$$

Using the Euler-Lagrange equation (10) to eliminate $\partial\mathcal{L}/\partial A^a{}_\gamma$, we get

$$\delta\mathcal{L} = \partial_\alpha \left[(\delta A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \right]\,.$$

Let us collect all terms which contribute to $\overline{\delta}\mathcal{L}$ in equation (17). We find (note that $\partial_\beta \delta x^\alpha = 0$ holds for all combinations of indices α, β)

$$0 = \overline{\delta}\mathcal{L} = \partial_\alpha \left[(\delta A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_{\gamma)}} + \delta x^\alpha \ \mathcal{L} \right] =$$

$$\partial_\alpha \left[(\partial_\gamma \epsilon^a(x) + C^a_{\ bc}\, \epsilon^c(x)\, A^b{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \right]$$

$$+\delta x_\beta\, \partial_\alpha \left[-(\partial^\beta A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} + g^{\alpha\beta}\, \mathcal{L} \right]$$

where equation (16) was used. To be able to factor δx_β also out of the first bracket on the right-hand side, one has to request

$$\epsilon^a(x) = \delta x_\beta\, B^{a\,\beta}(x) \tag{18}$$

where $B^{a\,\beta}(x)$ is a not yet determined gauge field. With this we get

$$\begin{aligned} 0 = \delta x_\beta\, \partial_\alpha \Bigg[& (\partial^\beta\, A^a{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \\ & - (\partial_\gamma\, B^{a\,\beta} + C^a_{\ bc}\, B^{c\,\beta}\, A^b{}_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} - g^{\alpha\beta}\, \mathcal{L} \Bigg]\,. \end{aligned} \tag{19}$$

Equation (11) implies that the contribution from the gauge transformations is a total divergence,

$$\begin{aligned} & \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)}\, B^{a\,\beta} \right) = \\ & \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)}\, \left[\partial_\gamma\, B^{a\,\beta} \ + C^a_{\ bc}\, B^{c\,\beta}\, A^b{}_\gamma \right]\,. \end{aligned} \tag{20}$$

As the variations δx_β in (19) are independent, the energy-momentum tensor

$$\begin{aligned} \theta^{\alpha\beta} = & \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)}\, \left[\partial^\beta A^a{}_\gamma - (\partial_\gamma\, B^{a\,\beta} + C^a_{\ bc}\, B^{c\,\beta}\, A^b{}_\gamma) \right] \\ & - g^{\alpha\beta}\, \mathcal{L} \end{aligned} \tag{21}$$

gives the conserved currents

$$\partial_\alpha\, \theta^{\alpha\beta} = 0\,. \tag{22}$$

We demand that $\theta^{\alpha\beta}$ is symmetric. The Lagrangian term $g^{\alpha\beta}\mathcal{L}$ is manifestly symmetric and we have to deal with the other contributions. We note that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} = -F^{a\,\alpha\gamma}$$

with $F^{a\,\alpha\gamma}$ given by equation (2). Therefore, the choice

$$B^{a\,\beta}(x) = A^{a\,\beta}(x) \tag{23}$$

leads to

$$\theta^{\alpha\beta} = F^{a\,\alpha\gamma}\, F^a{}_\gamma{}^{\ \beta} - g^{\alpha\beta}\, \mathcal{L} \tag{24}$$

where we used the anti-symmetry of the structure constant under interchange of b and c. The tensor (24) is symmetric because of

$$F^{a\,\alpha\gamma}\, F^a{}_\gamma{}^{\ \beta} = F^{a\,\beta\gamma}\, F^a{}_\gamma{}^{\ \alpha}\ .$$

In conclusion, I have derived the transformation behavior (6) by demanding that the energy-momentum tensor from Noether's theorem comes out symmetric. To the many arguments why gauge invariance is needed, this adds another one: It is needed to make the energy-momentum distribution under local translational variations symmetric.

Note added

After posting this manuscript Prof. Jackiw kindly informed me that my result is a special case of his work [7], see [8] for details. Prof. Hehl communicated that the use of 1-Forms leads directly to a symmetric energy-momentum tensor, see for instance [9].

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